

# ON $p$ -PARTS OF CHARACTER DEGREES AND CONJUGACY CLASS SIZES OF FINITE GROUPS

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**ABSTRACT.** Let  $G$  be a finite group and  $\text{Irr}(G)$  the set of irreducible complex characters of  $G$ . Let  $e_p(G)$  be the largest integer such that  $p^{e_p(G)}$  divides  $\chi(1)$  for some  $\chi \in \text{Irr}(G)$ . We show that  $|G : \mathbf{F}(G)|_p \leq p^{ke_p(G)}$  for a constant  $k$ . This settles a conjecture of A. Moretó [17, Conjecture 4].

We also study the related problems of the  $p$ -parts of conjugacy class sizes of finite groups.

## 1. INTRODUCTION

Let  $G$  be a finite group and  $P$  be a Sylow  $p$ -subgroup of  $G$ , it is reasonable to expect that the degrees of irreducible characters of  $G$  somehow restrict those of  $P$ . The Ito-Michler theorem proves that each ordinary irreducible character degree is coprime to  $p$  if and only if  $G$  has a normal abelian Sylow  $p$ -subgroup. Of course, this implies that  $|G : \mathbf{F}(G)|_p = 1$ .

Let  $\text{Irr}(G)$  be the set of irreducible complex characters of  $G$ , and  $e_p(G)$  be the largest integer such that  $p^{e_p(G)}$  divides  $\chi(1)$  for some  $\chi \in \text{Irr}(G)$ . Isaacs [14] showed that if  $G$  is solvable, then the derived length of a Sylow  $p$ -subgroup of  $G$  is bounded above by  $2e_p(G) + 1$ .

Let  $b(P)$  denote the largest degree of an irreducible character of  $P$ . [17, Conjecture 4] suggested that  $\log b(P)$  is bounded as a function of  $e_p(G)$ . Moretó and Wolf [18] have proven this for  $G$  solvable and even something a bit stronger, namely the logarithm to the base of  $p$  of the  $p$ -part of  $|G : \mathbf{F}(G)|$  is bounded in terms of  $e_p(G)$ . In fact, they showed that  $|G : \mathbf{F}(G)|_p \leq p^{19e_p(G)}$  for any solvable groups [18, Corollary B (i)], and  $|G : \mathbf{F}(G)|_p \leq p^{2e_p(G)}$  for odd order groups [18, Corollary B (iii)].

Recently, Lewis, Navarro and Wolf [13] showed the following. Assume that  $G$  is solvable and  $e_p(G) = 1$ , then  $|G : \mathbf{F}(G)|_p \leq p^2$ .

In this paper, we show that for arbitrary group,  $|G : \mathbf{F}(G)|_p \leq p^{ke_p(G)}$  for some constant  $k$ . This settles [17, Conjecture 4].

**Theorem A.** *Let  $G$  be a finite group and  $e_p(G)$  be the largest integer such that  $p^{e_p(G)}$  divides  $\chi(1)$  for some  $\chi \in \text{Irr}(G)$ .*

- (1) *If  $p \geq 5$ , then  $\log_p |G : \mathbf{F}(G)|_p \leq 7.5e_p(G)$ .*
- (2) *If  $p = 3$ , then  $\log_p |G : \mathbf{F}(G)|_p \leq 23e_p(G)$ .*
- (3) *If  $p = 2$ , then  $\log_p |G : \mathbf{F}(G)|_p \leq 20e_p(G)$ .*

Let  $G$  be a finite group. Let  $p$  be a prime and we denote the  $p$ -part of the group order  $|G|_p = p^n$ . An irreducible ordinary character of  $G$  is called  $p$ -defect 0 if and only if its degree is divisible by  $p^n$ . By [4, Theorem 4.18],  $G$  has a character of  $p$ -defect 0 if and only if  $G$  has a  $p$ -block of defect 0.

It is an interesting problem to give necessary and sufficient conditions for the existence of  $p$ -blocks of defect zero. If a finite group  $G$  has a character of  $p$ -defect 0, then  $O_p(G) = 1$  [4,

Corollary 6.9]. Unfortunately, the converse is not true. Zhang [25] and Fukushima [8, 9] provided various sufficient conditions where a finite group  $G$  has a block of defect zero.

Although the block of defect zero may not exist in general, one could try to find the smallest defect  $d(B)$  of a block  $B$  of  $G$ . It is not true in general that there exists a block  $B$  with  $d(B) \leq \lfloor \frac{n}{2} \rfloor$ , as  $G = A_7(p = 2)$  shows us. However, the counterexamples were only found for  $p = 2$  and  $p = 3$ . By work of Michler and Willems [16, 22] every simple group except possibly the alternating group has a block of defect zero for  $p \geq 5$ . The alternating group case was settled by Granville and Ono in [7] using number theory.

For solvable groups, one of the results along this line is given by [3, Theorem A]. In [3], Espuelas and Navarro bounded the smallest defect  $d(B)$  of a block  $B$  of  $G$  using the  $p$ -part of  $|G|$ . Using an orbit theorem [2, Theorem 3.1] of solvable linear groups of odd order, they showed the following result. Let  $G$  be a (solvable) group of odd order such that  $O_p(G) = 1$  and  $|G|_p = p^n$ , then  $G$  contains a  $p$ -block  $B$  such that  $d(B) \leq \lfloor \frac{n}{2} \rfloor$ . The bound is best possible, as shown by an example in [3].

Based on these, the following question raised by Espuelas and Navarro in [3] seems to be natural. If  $G$  is a finite group with  $O_p(G) = 1$  for some prime  $p \geq 5$ , and denote  $|G|_p = p^n$ , does  $G$  contain a block of defect less than or equal to  $\lfloor \frac{n}{2} \rfloor$ ?

Not much has been done on this problem. In a recent paper [24, Theorem B], the author obtained a partial result toward the previous question by showing the following. Let  $G$  be a finite solvable group, let  $p$  be a prime such that  $p \geq 5$  and  $O_p(G) = 1$ , and we denote  $|G|_p = p^n$ . Then  $G$  contains a  $p$ -block  $B$  such that  $d(B) \leq \lfloor \frac{3n}{5} \rfloor$ .

In this paper, we study a related result about the size of defect group of arbitrary finite groups, and obtain an upper bound.

**Theorem B.** *Let  $p$  be a prime, and  $G$  be a finite group such that  $O_p(G) = 1$ . We denote the  $p$  part of the group order  $|G|_p = p^n$ . Then  $G$  contains a  $p$ -block  $B$  such that  $d(B) \leq \lfloor \alpha n \rfloor$  for some constant  $\alpha$ .*

## 2. AN ORBIT THEOREM

Theorem A and Theorem B are proved using an orbit theorem of  $p$ -solvable groups.

We first fix some notation:

- (1) We use  $\mathbf{F}(G)$  to denote the Fitting subgroup of  $G$ . Let  $\mathbf{F}_0(G) \leq \mathbf{F}_1(G) \leq \mathbf{F}_2(G) \leq \dots \leq \mathbf{F}_n(G) = G$  denote the ascending Fitting series, i.e.  $\mathbf{F}_0(G) = 1$ ,  $\mathbf{F}_1(G) = \mathbf{F}(G)$  and  $\mathbf{F}_{i+1}(G)/\mathbf{F}_i(G) = \mathbf{F}(G/\mathbf{F}_i(G))$ .  $\mathbf{F}_i(G)$  is the  $i$ th ascending Fitting subgroup of  $G$ .
- (2) We use  $F^*(G)$  to denote the generalized Fitting subgroup of  $G$ .
- (3) We use  $O_\infty(G)$  to denote the maximal normal solvable subgroup of  $G$ .
- (4) We use  $\text{Irr}(G)$  to denote the set of all the irreducible characters of the group  $G$ .
- (5) We use  $\text{cl}(G)$  to denote the set of all the conjugacy classes of the group  $G$ .
- (6) Let  $G$  be a finite group, we denote  $cd(G) = \{\chi(1) \mid \chi \in \text{Irr}(G)\}$ .
- (7) Let  $G$  be a finite solvable group, we denote  $\text{dl}(G)$  to be the derived length of  $G$ .

We need the following results about simple groups.

**Lemma 2.1.** *Let  $A$  act faithfully and coprimely on a nonabelian simple group  $S$ . Then  $A$  has at least 2 regular orbits on  $\text{Irr}(S)$ .*

*Proof.* This is [19, Proposition 2.6]. □

**Lemma 2.2.** *Let  $G$  be a non-abelian finite simple group, then  $|cd(G)| \geq 4$ .*

*Proof.* This follows from [15, Theorem 12.15]. □

**Lemma 2.3.** *Let  $G$  be a non-abelian finite simple group, then  $C = \text{Out}(G)$  has a normal series of the form  $A \triangleleft B \triangleleft C$  where  $A$  is abelian,  $B/A$  is cyclic and  $C/B \cong 1, S_2$  or  $S_3$ .*

*Proof.* This is a standard result by the classification of finite simple groups. □

We now prove an orbit theorem about  $p$ -solvable groups.

**Proposition 2.4.** *Let  $p$  be a prime such that  $p \geq 3$ , and assume  $H$  is a  $p$ -solvable group such that  $\mathbf{F}(H) = 1$ . Suppose that  $V = F^*(H) = S_1 \times \cdots \times S_n$  where  $S_i \cong S$  for  $1 \leq i \leq n$  and  $S$  is a nonabelian simple group. Then  $H$  acts faithfully on  $V$ . Since  $V$  acts by inner automorphisms on  $V$  this implies that  $G = H/V$  embeds in  $\text{Out}(V) \cong \text{Out}(S) \wr \text{Sym}(n)$ . Moreover  $G$  acts on  $\text{Irr}(V)$ . We consider the action of  $G$  on  $\text{Irr}(V)$  and have the followings,*

- (1) *There exist four  $G$ -orbits with representatives  $v_1, v_2, v_3$  and  $v_4 \in \text{Irr}(V)$  of different degrees such that for any  $P \in \text{Syl}_p(G)$ , we have  $\mathbf{C}_P(v_j) \subseteq \text{Out}(S_1) \times \cdots \times \text{Out}(S_n)$  for  $1 \leq j \leq 4$ .*
- (2) *Assume  $p \geq 5$ , then there exists  $N \triangleleft G$ ,  $N \subseteq \mathbf{F}_2(G)$  and there exist four  $G$ -orbits with representatives  $v_1, v_2, v_3$  and  $v_4 \in \text{Irr}(V)$  of different degrees such that for any  $P \in \text{Syl}_p(G)$ , we have  $\mathbf{C}_P(v_j) \subseteq N$  for  $1 \leq j \leq 4$ . Moreover, the Sylow  $p$ -subgroup of  $\mathbf{NF}(G)/\mathbf{F}(G)$  is abelian.*
- (3) *Assume  $p = 3$ , then there exists  $N \triangleleft G$ ,  $N \subseteq \mathbf{F}_3(G)$  and there exist four  $G$ -orbits with representatives  $v_1, v_2, v_3$  and  $v_4 \in \text{Irr}(V)$  of different degrees such that for any  $P \in \text{Syl}_p(G)$ , we have  $\mathbf{C}_P(v_j) \subseteq N$  for  $1 \leq j \leq 4$ . Moreover, the Sylow  $p$ -subgroup of  $\mathbf{NF}_2(G)/\mathbf{F}_2(G)$  is abelian.*

*Proof.* (2) and (3) follow from (1) since  $C = \text{Out}(S)$  has a normal series of the form  $A \triangleleft B \triangleleft C$  where  $A$  is abelian,  $B/A$  is cyclic and  $C/B \cong 1, S_2$  or  $S_3$  by Lemma 2.3. For (2) we choose  $N = \mathbf{F}_2(G \cap (\text{Out}(S_1) \times \cdots \times \text{Out}(S_n)))$ . For (3) we choose  $N = \mathbf{F}_3(G \cap (\text{Out}(S_1) \times \cdots \times \text{Out}(S_n)))$ .

Thus, we only need to show (1).

We note that  $\text{Irr}(V) = \text{Irr}(S_1) \times \cdots \times \text{Irr}(S_n)$ . We denote  $\bar{G} = G/(G \cap (\text{Out}(S_1) \times \cdots \times \text{Out}(S_n)))$ , and clearly  $\bar{G}$  is a permutation group on a set of  $n$  elements.

Step 1. Assume the action of  $\bar{G}$  is not transitive, then the result follows easily by induction.

We recall some basic facts about the decompositions of transitive groups. Let  $\bar{G}$  be a transitive permutation group acting on a set  $\Delta$ ,  $|\Delta| = n$ . A system of imprimitivity is a partition of  $\Delta$ , invariant under  $\bar{G}$ . A primitive group has no non-trivial system of imprimitivity. Let  $(\Delta_1, \dots, \Delta_m)$  denote a system of imprimitivity of  $\bar{G}$  with maximal block-size  $b$  ( $1 \leq b < n$ ;  $b = 1$  if and only if  $\bar{G}$  is primitive;  $bm = n$ ).

Step 2. Assume  $n > 1$  and the action of  $\bar{G}$  is transitive, then either  $\bar{G}$  is imprimitive and  $b > 1$  or  $\bar{G}$  itself is primitive and  $b = 1$ .

Let  $\bar{J}_i$  be the set-wise stabilizer of  $\bar{G}$  on  $\Delta_i$ , i.e.  $\bar{J}_i = \text{Stab}_{\bar{G}}(\Delta_i)$ . Then the groups  $\bar{J}_i$ s are permutationally equivalent transitive groups of degree  $b$ . Let  $\bar{K} = \cap_i \bar{J}_i$ . We know that  $\bar{K}$  is a normal subgroup of  $\bar{G}$  stabilizing each of the blocks  $\Delta_i$ , and  $\bar{G}/\bar{K}$  is a primitive group of degree  $m$  acting upon the set of blocks  $\Delta_i$  and hence  $\{1, \dots, m\}$ . We denote the set  $\{1, \dots, m\}$  to be  $\Omega$ .

We define  $V_i = \prod_{t \in \Delta_i} S_t$ . Consider the action of  $\bar{J}_i$  on  $\text{Irr}(V_i) = \prod_{t \in \Delta_i} \text{Irr}(S_t)$ . If  $\bar{G}$  is imprimitive, then since  $1 < b < n$ , by induction there exist four  $\bar{J}_i$ -orbits with representatives  $\theta_i, \lambda_i, \chi_i$  and  $\psi_i \in \text{Irr}(V_i)$  such that for any  $\bar{P} \in \text{Syl}_p(\bar{J}_i)$ ,  $\mathbf{C}_{\bar{P}}(\theta_i) = \mathbf{C}_{\bar{P}}(\lambda_i) = \mathbf{C}_{\bar{P}}(\chi_i) = \mathbf{C}_{\bar{P}}(\psi_i) = 1$ . If  $\bar{G}$  is primitive, then  $b = 1$  and the same conclusion holds by Lemma 2.2.

It is clear that we may choose  $\theta_i, 1 \leq i \leq m$  to be conjugate by the action of  $\bar{G}/\bar{K}$  and we can do the same for  $\lambda_i, \chi_i$  and  $\psi_i, 1 \leq i \leq m$ . We also denote  $\theta(1) = \theta_i(1), \lambda(1) = \lambda_i(1), \chi(1) = \chi_i(1)$  and  $\psi(1) = \psi_i(1)$ . By re-indexing, we may assume that  $\theta(1) > \lambda(1) > \chi(1) > \psi(1)$ .

Assume  $p \nmid |\bar{G}/\bar{K}|$ , we set  $v_1 = \prod_{i=1}^m \theta_i, v_2 = \prod_{i=1}^m \lambda_i, v_3 = \prod_{i=1}^m \chi_i$  and  $v_4 = \prod_{i=1}^m \psi_i$ . It is clear that  $v_1(1) = \theta(1)^m > v_2(1) = \lambda(1)^m > v_3(1) = \chi(1)^m > v_4(1) = \psi(1)^m$ .

Assume  $p \mid |\bar{G}/\bar{K}|$  and  $m \geq 5$ , then since  $\bar{G}/\bar{K}$  is  $p$ -solvable, we know that  $\text{Alt}(m) \not\leq \bar{G}/\bar{K}$ . By [1, Lemma 1](b), we can find a partition  $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4$  such that  $\text{Stab}_{\bar{G}/\bar{K}}(\Omega_1) \cap \text{Stab}_{\bar{G}/\bar{K}}(\Omega_2) \cap \text{Stab}_{\bar{G}/\bar{K}}(\Omega_3) \cap \text{Stab}_{\bar{G}/\bar{K}}(\Omega_4)$  is a 2-group, and  $|\Omega_1|, |\Omega_2|, |\Omega_3|$  and  $|\Omega_4|$  are not all the same. We denote  $t_i = |\Omega_i|, 1 \leq i \leq 4$ . By re-indexing, we may assume that  $t_1 \geq t_2 \geq t_3 \geq t_4$ .

Since we know that not all the  $t_i$ s are the same, we must have one of the followings,

- (1)  $t_1 > t_2 \geq t_3 \geq t_4$ . In this case, we construct four irreducible characters,
  - (a)  $v_1 = \prod_{i=1}^m \alpha_i$ , where  $\alpha_i = \theta_i$  if  $i \in \Omega_1, \alpha_i = \lambda_i$  if  $i \in \Omega_2, \alpha_i = \chi_i$  if  $i \in \Omega_3, \alpha_i = \psi_i$  if  $i \in \Omega_4$ .
  - (b)  $v_2 = \prod_{i=1}^m \beta_i$ , where  $\beta_i = \lambda_i$  if  $i \in \Omega_1, \beta_i = \theta_i$  if  $i \in \Omega_2, \beta_i = \chi_i$  if  $i \in \Omega_3, \beta_i = \psi_i$  if  $i \in \Omega_4$ .
  - (c)  $v_3 = \prod_{i=1}^m \gamma_i$ , where  $\gamma_i = \chi_i$  if  $i \in \Omega_1, \gamma_i = \theta_i$  if  $i \in \Omega_2, \gamma_i = \lambda_i$  if  $i \in \Omega_3, \gamma_i = \psi_i$  if  $i \in \Omega_4$ .
  - (d)  $v_4 = \prod_{i=1}^m \delta_i$ , where  $\delta_i = \psi_i$  if  $i \in \Omega_1, \delta_i = \theta_i$  if  $i \in \Omega_2, \delta_i = \lambda_i$  if  $i \in \Omega_3, \delta_i = \chi_i$  if  $i \in \Omega_4$ .

Those four characters have different degrees since

$$v_1(1) = \theta(1)^{t_1} \lambda(1)^{t_2} \chi(1)^{t_3} \psi(1)^{t_4} > v_2(1) = \lambda(1)^{t_1} \theta(1)^{t_2} \chi(1)^{t_3} \psi(1)^{t_4} > \\ v_3(1) = \chi(1)^{t_1} \theta(1)^{t_2} \lambda(1)^{t_3} \psi(1)^{t_4} > v_4(1) = \psi(1)^{t_1} \theta(1)^{t_2} \lambda(1)^{t_3} \chi(1)^{t_4}.$$

- (2)  $t_1 = t_2 > t_3 \geq t_4$ . In this case, we construct four irreducible characters,
  - (a)  $v_1 = \prod_{i=1}^m \alpha_i$ , where  $\alpha_i = \theta_i$  if  $i \in \Omega_1, \alpha_i = \lambda_i$  if  $i \in \Omega_2, \alpha_i = \chi_i$  if  $i \in \Omega_3, \alpha_i = \psi_i$  if  $i \in \Omega_4$ .
  - (b)  $v_2 = \prod_{i=1}^m \beta_i$ , where  $\beta_i = \theta_i$  if  $i \in \Omega_1, \beta_i = \chi_i$  if  $i \in \Omega_2, \beta_i = \lambda_i$  if  $i \in \Omega_3, \beta_i = \psi_i$  if  $i \in \Omega_4$ .
  - (c)  $v_3 = \prod_{i=1}^m \gamma_i$ , where  $\gamma_i = \theta_i$  if  $i \in \Omega_1, \gamma_i = \psi_i$  if  $i \in \Omega_2, \gamma_i = \lambda_i$  if  $i \in \Omega_3, \gamma_i = \chi_i$  if  $i \in \Omega_4$ .
  - (d)  $v_4 = \prod_{i=1}^m \delta_i$ , where  $\delta_i = \chi_i$  if  $i \in \Omega_1, \delta_i = \psi_i$  if  $i \in \Omega_2, \delta_i = \lambda_i$  if  $i \in \Omega_3, \delta_i = \theta_i$  if  $i \in \Omega_4$ .

Those four characters have different degrees since

$$v_1(1) = \theta(1)^{t_1} \lambda(1)^{t_2} \chi(1)^{t_3} \psi(1)^{t_4} > v_2(1) = \theta(1)^{t_1} \chi(1)^{t_2} \lambda(1)^{t_3} \psi(1)^{t_4} > \\ v_3(1) = \theta(1)^{t_1} \psi(1)^{t_2} \lambda(1)^{t_3} \chi(1)^{t_4} > v_4(1) = \chi(1)^{t_1} \psi(1)^{t_2} \lambda(1)^{t_3} \theta(1)^{t_4}.$$

- (3)  $t_1 = t_2 = t_3 > t_4$ . In this case, we construct four irreducible characters,
  - (a)  $v_1 = \prod_{i=1}^m \alpha_i$ , where  $\alpha_i = \theta_i$  if  $i \in \Omega_1, \alpha_i = \lambda_i$  if  $i \in \Omega_2, \alpha_i = \chi_i$  if  $i \in \Omega_3, \alpha_i = \psi_i$  if  $i \in \Omega_4$ .
  - (b)  $v_2 = \prod_{i=1}^m \beta_i$ , where  $\beta_i = \theta_i$  if  $i \in \Omega_1, \beta_i = \lambda_i$  if  $i \in \Omega_2, \beta_i = \psi_i$  if  $i \in \Omega_3, \beta_i = \chi_i$  if  $i \in \Omega_4$ .

- (c)  $v_3 = \prod_{i=1}^m \gamma_i$ , where  $\gamma_i = \theta_i$  if  $i \in \Omega_1$ ,  $\gamma_i = \chi_i$  if  $i \in \Omega_2$ ,  $\gamma_i = \psi_i$  if  $i \in \Omega_3$ ,  $\gamma_i = \lambda_i$  if  $i \in \Omega_4$ .  
(d)  $v_4 = \prod_{i=1}^m \delta_i$ , where  $\delta_i = \lambda_i$  if  $i \in \Omega_1$ ,  $\delta_i = \chi_i$  if  $i \in \Omega_2$ ,  $\delta_i = \psi_i$  if  $i \in \Omega_3$ ,  $\delta_i = \theta_i$  if  $i \in \Omega_4$ .

Those four characters have different degrees since

$$v_1(1) = \theta(1)^{t_1} \lambda(1)^{t_2} \chi(1)^{t_3} \psi(1)^{t_4} > v_2(1) = \theta(1)^{t_1} \lambda(1)^{t_2} \psi(1)^{t_3} \chi(1)^{t_4} > \\ v_3(1) = \theta(1)^{t_1} \chi(1)^{t_2} \psi(1)^{t_3} \lambda(1)^{t_4} > v_4(1) = \lambda(1)^{t_1} \chi(1)^{t_2} \psi(1)^{t_3} \theta(1)^{t_4}.$$

Assume  $p \mid |\bar{G}/\bar{K}|$ ,  $m = 4$  and  $p = 3$ . We set  $v_1 = \theta_1 \lambda_2 \chi_3 \psi_4$ ,  $v_2 = \theta_1 \lambda_2 \psi_3 \psi_4$ ,  $v_3 = \theta_1 \chi_2 \psi_3 \psi_4$  and  $v_4 = \lambda_1 \chi_2 \psi_3 \psi_4$ . They have different degrees since  $v_1(1) = \theta(1) \lambda(1) \chi(1) \psi(1) > v_2(1) = \theta(1) \lambda(1) \psi^2(1) > v_3(1) = \theta(1) \chi(1) \psi^2(1) > v_4(1) = \lambda(1) \chi(1) \psi^2(1)$ .

It is clear that  $\mathbf{C}_{\bar{G}/\bar{K}}(v_1)$ ,  $\mathbf{C}_{\bar{G}/\bar{K}}(v_2)$ ,  $\mathbf{C}_{\bar{G}/\bar{K}}(v_3)$  and  $\mathbf{C}_{\bar{G}/\bar{K}}(v_4)$  is a 2-group.

Assume  $p \mid |\bar{G}/\bar{K}|$ ,  $m = 3$  and  $p = 3$ . We set  $v_1 = \theta_1 \lambda_2 \chi_3$ ,  $v_2 = \theta_1 \lambda_2 \psi_3$ ,  $v_3 = \theta_1 \chi_2 \psi_3$  and  $v_4 = \lambda_1 \chi_2 \psi_3$ . They have different degrees since  $v_1(1) = \theta(1) \lambda(1) \chi(1) > v_2(1) = \theta(1) \lambda(1) \psi(1) > v_3(1) = \theta(1) \chi(1) \psi(1) > v_4(1) = \lambda(1) \chi(1) \psi(1)$ .

It is clear that  $\mathbf{C}_{\bar{G}/\bar{K}}(v_1)$ ,  $\mathbf{C}_{\bar{G}/\bar{K}}(v_2)$ ,  $\mathbf{C}_{\bar{G}/\bar{K}}(v_3)$  and  $\mathbf{C}_{\bar{G}/\bar{K}}(v_4)$  is a 2-group.

Thus, we may always find four irreducible characters  $v_1, v_2, v_3$  and  $v_4$  in  $\text{Irr}(V)$  of different degrees such that for any  $P \in \text{Syl}_p(G)$ , we have  $\mathbf{C}_P(v_j) \subseteq K$  and thus  $\mathbf{C}_P(v_j) \subseteq \text{Out}(S)^n$  for  $1 \leq j \leq 4$ .

Step 3. Assume  $n = 1$ , then the result follows by Lemma 2.2.  $\square$

**Proposition 2.5.** *Let  $G$  be a finite  $p$ -solvable group where  $O_\infty(G) = 1$ . Then  $F^*(G) = E_1 \times \cdots \times E_m$  is a product of  $m$  finite non-abelian simple groups  $E_j$ ,  $1 \leq j \leq m$  permuted by  $G$ . We denote  $\bar{G} = G/F^*(G)$ .*

- (1) *Assume  $p \geq 5$ , then there exists  $N \triangleleft \bar{G}$  where  $N \subseteq \mathbf{F}_2(\bar{G})$ , and there exists  $v \in \text{Irr}(F^*(G))$  such that for any  $P \in \text{Syl}_p(\bar{G})$ , we have  $\mathbf{C}_P(v) \subseteq N$ . Moreover, the Sylow  $p$ -subgroup of  $N\mathbf{F}(\bar{G})/\mathbf{F}(\bar{G})$  is abelian.*
- (2) *Assume  $p = 3$ , then there exists  $N \triangleleft \bar{G}$  where  $N \subseteq \mathbf{F}_3(\bar{G})$ , and there exists  $v \in \text{Irr}(F^*(G))$  such that for any  $P \in \text{Syl}_p(\bar{G})$ , we have  $\mathbf{C}_P(v) \subseteq N$ . Moreover, the Sylow  $p$ -subgroup of  $N\mathbf{F}_2(\bar{G})/\mathbf{F}_2(\bar{G})$  is abelian.*

*Proof.* Clearly  $G$  acts faithfully on  $F^*(G)$ . Since  $F^*(G)$  acts by inner automorphisms on  $F^*(G)$ , this implies that  $\bar{G}$  embeds in  $\text{Out}(F^*(G))$ . Moreover  $\bar{G}$  acts on  $\text{Irr}(F^*(G))$ . Next, we group the simple groups in the direct product of  $F^*(G)$  where  $G$  acts transitively. We denote  $F^*(G) = L_1 \times \cdots \times L_s$ ,  $L_i = E_{i1} \times E_{i2} \times \cdots \times E_{im_i}$ ,  $1 \leq i \leq s$  where  $G$  transitively permutes the simple groups inside the direct product of  $L_i$ . Clearly  $E_{i1} \cong E_{i2} \cdots \cong E_{im_i}$ .

We see that  $\bar{G}$  can be embedded as a subgroup of  $\text{Out}(L_1) \times \cdots \times \text{Out}(L_s)$  and we denote  $K_i$  to be the image of  $\bar{G}$  in  $\text{Out}(L_i)$ .

If  $p \geq 5$ , by applying Proposition 2.4(1) to the action of  $K_i$  on  $\text{Irr}(L_i)$ , there exists  $v_i \in \text{Irr}(L_i)$ , and  $N_i \triangleleft K_i$  such that for any  $P_i \in \text{Syl}_p(K_i)$ ,  $\mathbf{C}_{P_i}(v_i) \subseteq N_i \subseteq \mathbf{F}_2(K_i)$ . Also the Sylow  $p$ -subgroup of  $N_i\mathbf{F}(K_i)/\mathbf{F}(K_i)$  is abelian.

Let  $v = \sum v_i$  and  $N = K \cap (\prod N_i)$ . Let  $P \in \text{Syl}_p(G)$ , and  $P_i$  to be the image of  $P$  in  $\text{Irr}(V_i)$ . Then  $\mathbf{C}_P(v) \subseteq \prod \mathbf{C}_{P_i}(v_i) \subseteq \prod N_i$ . Clearly  $N\mathbf{F}(K)/\mathbf{F}(K) \subseteq \prod N_i\mathbf{F}(K_i)/\mathbf{F}(K_i)$  and the result follows.

If  $p = 3$ , by applying Proposition 2.4(2) to the action of  $K_i$  on  $\text{Irr}(L_i)$ , there exists  $v_i \in \text{Irr}(L_i)$ , and  $N_i \triangleleft K_i$  such that for any  $P_i \in \text{Syl}_p(K_i)$ ,  $\mathbf{C}_{P_i}(v_i) \subseteq N_i \subseteq \mathbf{F}_3(K_i)$ . Also the Sylow  $p$ -subgroup of  $N_i \mathbf{F}_2(K_i)/\mathbf{F}_2(K_i)$  is abelian.

Let  $v = \sum v_i$  and  $N = K \cap (\prod N_i)$ . Let  $P \in \text{Syl}_p(G)$ , and  $P_i$  to be the image of  $P$  in  $\text{Irr}(V_i)$ . Then  $\mathbf{C}_P(v) \subseteq \prod \mathbf{C}_{P_i}(v_i) \subseteq \prod N_i$ . Clearly  $N \mathbf{F}_3(K)/\mathbf{F}_3(K) \subseteq \prod N_i \mathbf{F}_3(K_i)/\mathbf{F}_3(K_i)$  and the result follows.  $\square$

### 3. $p$ PART OF $|G : \mathbf{F}(G)|$ , CHARACTER DEGREES AND CONJUGACY CLASS SIZES

We now prove Theorem A and some related results in this section. We first obtain bounds for  $p$ -solvable groups and then extend those to arbitrary groups.

**Theorem 3.1.** *Let  $G$  be a finite  $p$ -solvable group where  $p$  is a prime and  $p \geq 5$ . Suppose that  $p^{a+1}$  does not divide  $\chi(1)$  for all  $\chi \in \text{Irr}(G)$  and let  $P \in \text{Syl}_p(G)$ , then  $|G : \mathbf{F}(G)|_p \leq p^{5.5a}$ ,  $b(P) \leq p^{6.5a}$  and  $\text{dl}(P) \leq 5 + \log_2 a + \log_2 6.5$ .*

*Proof.* Let  $T = O_\infty(G)$ , the maximal normal subgroup of  $G$ . Since  $\mathbf{F}(G) \subseteq T$ ,  $\mathbf{F}(T) = \mathbf{F}(G)$ . Since  $T \triangleleft G$ ,  $p^{a+1}$  does not divide  $\lambda(1)$  for all  $\lambda \in \text{Irr}(T)$ . Thus by [24, Remark of Corollary 5.3],  $|T : \mathbf{F}(G)|_p \leq p^{2.5a}$ .

Let  $\tilde{G} = G/T$  and  $\bar{G} = \tilde{G}/F^*(\tilde{G})$ . It is clear that  $F^*(\tilde{G})$  is a direct product of finite non-abelian simple groups. By Proposition 2.5(1), there exists  $v \in \text{Irr}(F^*(\tilde{G}))$ ,  $N \triangleleft \bar{G}$  where  $N \subseteq \mathbf{F}_2(\bar{G})$  such that for any  $P \in \text{Syl}_p(\bar{G})$ , we have  $\mathbf{C}_P(v) \subseteq N$ , and the Sylow  $p$ -subgroup of  $N \mathbf{F}(\bar{G})/\mathbf{F}(\bar{G})$  is abelian. It is clear that we may find  $\tilde{\gamma} \in \text{Irr}(\tilde{G})$  such that  $|\tilde{G} : N|_p$  divides  $\tilde{\gamma}(1)$ .

Let  $P/\mathbf{F}(\bar{G})$  be a Sylow  $p$ -subgroup of  $N \mathbf{F}(\bar{G})/\mathbf{F}(\bar{G})$ . Where  $Y = O_{p'}(\mathbf{F}(\bar{G}))$ , observe that  $W = \text{Irr}(Y/\Phi(Y))$  is a faithful and completely reducible  $P/\mathbf{F}(\bar{G})$ -module. By Gow's regular orbit theorem [6, 2.6], we have  $\mu \in W$  such that  $\mathbf{C}_P(\mu) = \mathbf{F}(\bar{G})$ . We may view  $\mu$  as a character of the preimage  $X$  of  $Y$  in  $\bar{G}$ . Take  $\bar{\alpha} \in \text{Irr}(\bar{G})$  lying over  $\mu$ , and  $\bar{\alpha}$  lies over an irreducible character  $\bar{\psi}$  of  $P$  lying over  $\mu$ . Clearly,  $\bar{\psi}(1)_p \geq |N \mathbf{F}(\bar{G})/\mathbf{F}(\bar{G})|_p$ . As  $P$  is normal in  $G$ , we have  $\bar{\alpha}(1)_p \geq \bar{\psi}(1)_p \geq |N \mathbf{F}(\bar{G})/\mathbf{F}(\bar{G})|_p$ .

Let  $P_1$  be the Sylow  $p$ -subgroup of  $\mathbf{F}(\bar{G}) \cap N$ . By Lemma 2.1, we may find  $\nu \in \text{Irr}(F^*(\tilde{G}))$  such that  $\mathbf{C}_{P_1}(\nu) = 1$ . Thus by a similar argument as before, we may find an irreducible character  $\tilde{\beta}$  of  $\tilde{G}$  such that  $\tilde{\beta}(1)_p \geq |N \cap \mathbf{F}(\bar{G})|_p$ .

Thus  $|G : \mathbf{F}(G)|_p \leq p^{5.5a}$ . If  $P \in \text{Syl}_p(G)$ , then  $b(P) \leq |P : O_p(G)| |b(O_p(G))| = |G : \mathbf{F}(G)|_p |b(O_p(G))| \leq p^{5.5a} p^a = p^{6.5a}$ .

Now, we want to prove the last part of the statement. By [15, Theorem 12.26] and the nilpotency of  $P$ , we have that  $P$  has an abelian subgroup  $B$  of index at most  $b(P)^4$ . By [20, Theorem 5.1], we deduce that  $P$  has a normal abelian subgroup  $A$  of index at most  $|P : B|^2$ . Thus,  $|P : A| \leq |P : B|^2 \leq b(P)^{8s}$ , where  $b(P) = p^s$ . By [11, Satz III.2.12],  $\text{dl}(P/A) \leq 1 + \log_2(8s)$  and so  $\text{dl}(P) \leq 2 + \log_2(8s) = 5 + \log_2(s)$ . Since  $s$  is at most  $6.5a$ , the result follows.  $\square$

We now state the conjugacy analogs of Theorem 3.1. Given a group  $G$ , we write  $b^*(G)$  to denote the largest size of the conjugacy classes of  $G$ .

**Theorem 3.2.** *Let  $G$  be a  $p$ -solvable group where  $p$  is a prime and  $p \geq 5$ . Suppose that  $p^{a+1}$  does not divide  $|C|$  for all  $C \in \text{cl}(G)$  and let  $P \in \text{Syl}_p(G)$ , then  $|G : \mathbf{F}(G)|_p \leq p^{5.5a}$ ,  $b^*(P) \leq p^{6.5a}$  and  $|P'| \leq p^{6.5a(6.5a+1)/2}$ .*

*Proof.* The proof of the first statement goes similarly as the previous one. Write  $N = O_p(G)$ . It is clear that  $|N : \mathbf{C}_N(x)|$  divides  $|G : \mathbf{C}_G(x)|$  for all  $x \in G$ . Thus, if we take  $x \in P$  we have that

$$|\text{cl}_P(x)| = |P : \mathbf{C}_P(x)| \leq |P : N| |N : \mathbf{C}_N(x)| \leq p^{5.5a} p^a = p^{6.5a}$$

Finally, to obtain the bounds for the order of  $P'$  it suffices to apply a theorem of Vaughan-Lee [12, Theorem VIII.9.12].  $\square$

**Theorem 3.3.** *If  $G$  is solvable, there exists a product  $\theta = \chi_1 \dots \chi_t$  of distinct irreducible characters  $\chi_i$  of  $G$  such that  $|G : \mathbf{F}(G)|$  divides  $\theta(1)$  and  $t \leq 15$ . Furthermore, if  $|\mathbf{F}_8(G)|$  is odd then we can take  $t \leq 3$  and if  $|G|$  is odd we can take  $t \leq 2$ .*

*Proof.* This is [23, Theorem 5.1]. The statement in that paper should be  $t \leq 15$  instead of  $t \leq 19$ , and we take the opportunity to correct it here.  $\square$

**Theorem 3.4.** *If  $G$  is solvable, there exists conjugacy classes  $C_1, \dots, C_t$  such that  $|G : \mathbf{F}(G)|$  divides  $|C_1| \dots |C_t|$  and  $t \leq 15$ . Furthermore, if  $|\mathbf{F}_8(G)|$  is odd then we can take  $t \leq 3$  and if  $|G|$  is odd we can take  $t \leq 2$ .*

*Proof.* This is the conjugacy class version of Theorem 3.3.  $\square$

For the case  $p = 3$ , we have the following result.

**Theorem 3.5.** *Let  $G$  be a  $p$ -solvable group where  $p = 3$ . Suppose that  $p^{a+1}$  does not divide  $\chi(1)$  for all  $\chi \in \text{Irr}(G)$  and let  $P \in \text{Syl}_p(G)$ , then  $|G : \mathbf{F}(G)|_p \leq p^{20a}$ ,  $b(P) \leq p^{21a}$  and  $\text{dl}(P) \leq 5 + \log_2 a + \log_2 21$ .*

*Proof.* The proof is similar to the proof of Theorem 3.1 but using Theorem 3.3 instead of [24, Remark of Corollary 5.3], and Proposition 2.5(2) instead of Proposition 2.5(1).  $\square$

**Theorem 3.6.** *Let  $G$  be a  $p$ -solvable group where  $p = 3$ . Suppose that  $p^{a+1}$  does not divide  $|C|$  for all  $C \in \text{cl}(G)$  and let  $P \in \text{Syl}_p(G)$ , then  $|G : \mathbf{F}(G)|_p \leq p^{20a}$ ,  $b^*(P) \leq p^{21a}$  and  $|P'| \leq p^{21a(21a+1)/2}$ .*

*Proof.* This is the conjugacy class version of Theorem 3.5.  $\square$

**Lemma 3.7.** *Let  $S$  be a finite simple group and  $p$  be a prime divisor of  $|S|$ .*

- (1) *If  $p \geq 5$ , then there exist  $\chi \in \text{Irr}(S)$  such that  $v_p(|\text{Aut}(S)|/\chi^2(1)) < 0$ .*
- (2) *If  $p = 3$ , then there exist  $\chi \in \text{Irr}(S)$  such that  $v_p(|\text{Aut}(S)|/\chi^3(1)) < 0$ .*
- (3) *If  $p = 2$ , then there exist  $\chi \in \text{Irr}(S)$  such that  $v_p(|\text{Aut}(S)|/\chi^5(1)) < 0$ .*

*Proof.* This follows from [10, Lemma 2.1].  $\square$

**Lemma 3.8.** *Let  $S$  be a finite simple group and  $p$  be a prime divisor of  $|S|$ .*

- (1) *If  $p \geq 5$ , then there exist  $C \in \text{cl}(S)$  such that  $v_p(|\text{Aut}(S)|/|C|^2) < 0$ .*
- (2) *If  $p = 3$ , then there exist  $C \in \text{cl}(S)$  such that  $v_p(|\text{Aut}(S)|/|C|^2) < 0$ .*
- (3) *If  $p = 2$ , then there exist  $C \in \text{cl}(S)$  such that  $v_p(|\text{Aut}(S)|/|C|^2) < 0$ .*

*Proof.* For the simple group of Lie type and any prime  $p$  or the alternating group and  $p \geq 5$ , there is a  $p$ -block of defect 0. Hence there is a conjugacy class  $\text{cl}_G(x)$  of  $p$ -defect 0, thus  $|G|_p$  divides  $|\text{cl}_G(x)|$ . And the result follows by the proof of [10, Lemma 1.2].

Thus one only needs to consider the alternating groups and  $p = 2, 3$ .

Set  $\alpha = (12 \cdots n)$  if  $n$  is odd, we get that  $\alpha \in A_n$  and  $|\text{cl}_{S_n}(\alpha)| = (n-1)!$ . Thus  $|\text{cl}_{A_n}(\alpha)|$  is a multiple of  $\frac{1}{2}(n-1)!$  and the result is clear.

Set  $\alpha = (12 \cdots n-1)$  if  $n$  is even, we get that  $\alpha \in A_n$  and  $|\text{cl}_{S_n}(\alpha)| = \frac{n!}{n-1}$ . Thus  $|\text{cl}_{A_n}(\alpha)|$  is a multiple of  $\frac{1}{2} \cdot \frac{n!}{n-1}$  and the result is clear.  $\square$

**Hypothesis 3.9.** Let  $p$  be a prime and let  $N = W_1 \times \cdots \times W_s$  be a normal subgroup of a finite group  $G$  with the following assumptions:  $\mathbf{C}_G(N) = 1$ ; every  $W_i$ ,  $1 \leq i \leq s$ , is a nonabelian simple group of order divisible by  $p$ .

**Lemma 3.10.** Let  $G, N, p$  be as in Hypothesis 3.9. If there exists  $\phi_i \in \text{Irr}(W_i)$  such that  $v_p(|\text{Aut}(W_i)|/\phi_i(1)^k) < 0$  for every  $i = 1, \dots, s$ , then there exist  $\phi \in \text{Irr}(N)$  such that  $v_p(|G|/\phi(1)^k) < 0$ .

*Proof.* The proof is the same as [21, Lemma 2.6].  $\square$

**Lemma 3.11.** Let  $G, N, p$  be as in Hypothesis 3.9. If there exists  $C_i \in \text{cl}(W_i)$  such that  $v_p(|\text{Aut}(W_i)|/|C_i|^k) < 0$  for every  $i = 1, \dots, s$ , then there exist  $C \in \text{cl}(N)$  such that  $v_p(|G|/|C|^k) < 0$ .

*Proof.* The proof is the same as [21, Lemma 2.6].  $\square$

**Theorem 3.12.** Let  $G$  be a finite group and  $p$  be a prime. Suppose that  $p^{a+1}$  does not divide  $\chi(1)$  for all  $\chi \in \text{Irr}(G)$  and let  $P \in \text{Syl}_p(G)$ .

- (1) If  $p \geq 5$ , then  $|G : \mathbf{F}(G)|_p \leq p^{7.5a}$ ,  $b(P) \leq p^{8.5a}$  and  $\text{dl}(P) \leq 5 + \log_2 a + \log_2 8.5$ .
- (2) If  $p = 3$ , then  $|G : \mathbf{F}(G)|_p \leq p^{23a}$ ,  $b(P) \leq p^{24a}$  and  $\text{dl}(P) \leq 5 + \log_2 a + \log_2 24$ .
- (3) If  $p = 2$ , then  $|G : \mathbf{F}(G)|_p \leq p^{20a}$ ,  $b(P) \leq p^{20a}$  and  $\text{dl}(P) \leq 5 + \log_2 a + \log_2 20$ .

*Proof.* Let  $T$  be the maximal normal  $p$ -solvable subgroup of  $G$ . Since  $\mathbf{F}(G) \subseteq T$ ,  $\mathbf{F}(T) = \mathbf{F}(G)$ . Since  $T \triangleleft G$ ,  $p^{a+1}$  does not divide  $\lambda(1)$  for all  $\lambda \in \text{Irr}(T)$ .

If  $p \geq 5$ , then  $|T : \mathbf{F}(G)|_p \leq p^{5.5a}$  by Theorem 3.1.

If  $p = 3$ , then  $|T : \mathbf{F}(G)|_p \leq p^{20a}$  by Theorem 3.5.

If  $p = 2$ , then  $|T : \mathbf{F}(G)|_p \leq p^{15a}$  by Theorem 3.3.

We now consider  $\bar{G} = G/T$ , we know that  $F^*(\bar{G})$  is a direct product of the non-abelian simple groups, where  $p$  divides the order of each of them.

Since  $\bar{G}$  and  $F^*(\bar{G})$  satisfy Hypothesis 3.9, by Lemma 3.10 and Lemma 3.7, we have that

$|\bar{G}|_p \leq p^{2a}$  if  $p \geq 5$ .

$|\bar{G}|_p \leq p^{3a}$  if  $p = 3$ .

$|\bar{G}|_p \leq p^{5a}$  if  $p = 2$ .

Thus, we have,

- (1)  $|G : \mathbf{F}(G)|_p \leq |G : T|_p |T : \mathbf{F}(G)|_p \leq p^{7.5a}$  if  $p \geq 5$ .
- (2)  $|G : \mathbf{F}(G)|_p \leq |G : T|_p |T : \mathbf{F}(G)|_p \leq p^{23a}$  if  $p = 3$ .
- (3)  $|G : \mathbf{F}(G)|_p \leq |G : T|_p |T : \mathbf{F}(G)|_p \leq p^{20a}$  if  $p = 2$ .

The bounds for  $b(P)$  and  $\text{dl}(P)$  follow from the same arguments as in Theorem 3.1.  $\square$

**Theorem 3.13.** Let  $G$  be a finite group where  $p$  is a prime. Suppose that  $p^{a+1}$  does not divide  $|C|$  for all  $C \in \text{cl}(G)$  and let  $P \in \text{Syl}_p(G)$ .



- (1) If  $p \geq 5$ , then  $|G : \mathbf{F}(G)|_p \leq p^{7.5a}$ ,  $b^*(P) \leq p^{8.5a}$  and  $|P'| \leq p^{8.5a(8.5a+1)/2}$ .
- (2) If  $p = 3$ , then  $|G : \mathbf{F}(G)|_p \leq p^{22a}$ ,  $b^*(P) \leq p^{23a}$  and  $|P'| \leq p^{23a(23a+1)/2}$ .
- (3) If  $p = 2$ , then  $|G : \mathbf{F}(G)|_p \leq p^{17a}$ ,  $b^*(P) \leq p^{18a}$  and  $|P'| \leq p^{18a(18a+1)/2}$ .

*Proof.* Let  $T$  be the maximal normal  $p$ -solvable subgroup of  $G$ . Since  $\mathbf{F}(G) \subseteq T$ ,  $\mathbf{F}(T) = \mathbf{F}(G)$ . Since  $T \triangleleft G$ ,  $p^{a+1}$  does not divide  $|C|$  for all  $C \in \text{cl}(T)$ .

If  $p \geq 5$ , then  $|T : \mathbf{F}(G)|_p \leq p^{5.5a}$  by Theorem 3.2.

If  $p = 3$ , then  $|T : \mathbf{F}(G)|_p \leq p^{20a}$  by Theorem 3.6.

If  $p = 2$ , then  $|T : \mathbf{F}(G)|_p \leq p^{15a}$  by Theorem 3.4.

We now consider  $\bar{G} = G/T$ , we know that  $F^*(\bar{G})$  is a direct product of the non-abelian simple groups, where  $p$  divides the order of each of them.

Since  $\bar{G}$  and  $F^*(\bar{G})$  satisfy Hypothesis 3.9, by Lemma 3.11 and Lemma 3.8, we have that  $|\bar{G}|_p \leq p^{2a}$ .

Thus, we have,

- (1)  $|G : \mathbf{F}(G)|_p \leq |G : T|_p |T : \mathbf{F}(G)|_p \leq p^{7.5a}$  if  $p \geq 5$ .
- (2)  $|G : \mathbf{F}(G)|_p \leq |G : T|_p |T : \mathbf{F}(G)|_p \leq p^{22a}$  if  $p = 3$ .
- (3)  $|G : \mathbf{F}(G)|_p \leq |G : T|_p |T : \mathbf{F}(G)|_p \leq p^{17a}$  if  $p = 2$ .

The bounds for  $b^*(P)$  and  $|P'|$  follow from the same arguments as in Theorem 3.2.  $\square$

#### 4. BLOCKS AND DEFECT GROUPS

Now we are ready to prove Theorem B, which we restate.

**Theorem B.** *Let  $p$  be a prime, and  $G$  be a finite group such that  $O_p(G) = 1$ . We denote the  $p$  part of the group order  $|G|_p = p^n$ . Then  $G$  contains a  $p$ -block  $B$  such that  $d(B) \leq \lfloor \alpha n \rfloor$  where,*

- (1)  $\alpha = \frac{6.5}{7.5}$  if  $p \geq 5$ .
- (2)  $\alpha = \frac{22}{23}$  if  $p = 3$ .
- (3)  $\alpha = \frac{19}{20}$  if  $p = 2$ .

*Proof.* By the proof of Theorem 3.12, there exists a  $\chi \in \text{Irr}(G)$  such that  $(\chi(1)_p)^\beta \geq |G|_p$ , i.e.  $\chi(1)_p \geq p^{\frac{n}{\beta}}$ .

where

- (1)  $\beta = 7.5$  if  $p \geq 5$ .
- (2)  $\beta = 23$  if  $p \geq 3$ .
- (3)  $\beta = 20$  if  $p \geq 2$ .

If  $|G|_p = p^n$  and  $B$  be a  $p$ -block with  $d(B) = d$ , then  $p^{n-d}$  is the largest  $p$ -part for all irreducible characters in  $\text{Irr}(G) \cap B$ .

If  $G$  has an irreducible character  $\chi$  of degree divisible by  $p^t$ , then  $d(B) \leq n - t$  where  $B$  is a  $p$ -block which  $\chi$  lies in.

Thus we know there is a  $p$ -block such that  $B$  such that  $d(B) \leq n - \frac{n}{\beta} \leq \frac{(\beta-1)n}{\beta}$ .  $\square$

**Remark:** In order to improve the bounds of the results in this paper, one might need to study the situation when a  $p$ -solvable group acts on a field of characteristic does not equal to  $p$ , and hope that a similar result as [24, Theorem 3.3] holds. Also, a strengthened version of Lemma 2.1 would also be helpful in improving the bounds.

## 5. ACKNOWLEDGEMENT

The first author would like to thank for the financial support from the AMS-Simons travel grant. The second author is partially supported by a grant from the NSF of China (11471054).

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